+359/1/48+

EPFL

359

Teacher : Prof. Emmanuel Abbe
MATH-232 Probability and Statistics - Final Exam
20 June 2023
Duration : 180 minutes

Additional

SCIPER : **0**

Do not turn the page before the start of the exam. This document is double-sided, has 24 pages, the last ones possibly blank. Do not unstaple.

- Place your student card on your table.
- **No other paper materials** are allowed to be used during the exam.
- Using a **calculator** or any electronic device is **not permitted** during the exam.
- A cheat sheet is provided on the last pages of this booklet.
- For the **multiple choice** questions, we give :
  +3 points if your answer is correct,
  0 points if you give no answer or more than one,
  −1 points if your answer is incorrect.
- For the **true/false** questions, we give :
  +2 points if your answer is correct,
  0 points if you give no answer or more than one,
  −2 points if your answer is incorrect.
- Use a **black or dark blue ballpen** and clearly erase with **correction fluid** if necessary.
- If a question is wrong, the teacher may decide to nullify it.
- The multiple choice questions are shuffled and hence are not in the order of difficultly.

Respectez les consignes suivantes | Observe this guidelines | Beachten Sie bitte die unten stehenden Richtlinien		
choisir une réponse | select an answer Antwort auswählen	ne PAS choisir une réponse | NOT select an answer NICHT Antwort auswählen	Corriger une réponse | Correct an answer Antwort korrigieren
ce qu'il ne faut **PAS** faire | what should **NOT** be done | was man **NICHT** tun sollte		

First part: multiple choice questions

For each question, mark the box corresponding to the correct answer. Each question has **exactly one** correct answer. No justifications are needed for this part.

Question 1 Suppose that random variable U follows the uniform distribution on $[0,1]$, i.e., $U \sim \text{Uniform}(0,1)$. Define $X = -\lambda \ln U$. What is $\mathbb{P}(X > x)$? ($\exp(z) = e^z$)

- ☐ $\frac{1}{\lambda}\exp(-x/\lambda)$
- ■ $\exp(-x/\lambda)$
- ☐ $\lambda\exp(-\lambda x)$
- ☐ $\exp(-\lambda x)$

Question 2 Let $X_1, X_2, \ldots$ be a sequence of independent Poisson random variables such that $X_n \sim \text{Poisson}(n\lambda)$ where $\lambda > 0$ is a constant. Consider the sequence given by $Y_n = \frac{X_n}{n}$ and the following claims:

(a) Y_n converges to λ in distribution.

(b) Y_n converges to λ in probability.

(c) Y_n converges to λ in mean square.

How many of the claims above are actually valid? In other words, how many modes of convergence (among in distribution, in probability and in mean square) hold for $Y_n \to \lambda$?

- ■ 3
- ☐ 1
- ☐ 2
- ☐ 0

Question 3 Consider three bits $b_1, b_2, b_3 \in \{0,1\}$ that are sent over a noisy channel that flips each bit independently with probability $p < \frac{1}{2}$. Assume that one transmits two possible sequences on this channel with equal probability: either 000 or 111. Therefore, we define the null and alternative hypotheses as H_0 for '000 is transmitted' and H_1 for '111 is transmitted'. What is the optimal average error probability of a hypothesis test (again, each hypothesis is selected with probability 1/2)?

- ■ $p^2(3-2p)$
- ☐ $p^2(1-p)$
- ☐ p^3
- ☐ p^2

Question 4 Consider a random variable θ taken uniformly over $[0, 2\pi]$, i.e.,

$$f_\theta(\theta) = \begin{cases} \frac{1}{2\pi} & 0 \le \theta \le 2\pi \\ 0 & \text{otherwise} \end{cases}.$$

Note that this is equivalent with sampling a point from the unit circle randomly where θ is the angle. We define (X, Y) as the coordinates of the random point from the circle, i.e., $X = \cos\theta$ and $Y = \sin\theta$. What is $\mathbb{E}[X^2]$?
(Hint: This question is only about X but one may still use Y to come up with the answer.)

- ■ $\frac{1}{2}$
- ☐ $\frac{3}{4}$
- ☐ $\frac{2}{3}$
- ☐ $\frac{1}{3}$

Question 5 Suppose that an individual has Covid with probability 0.01. Further, consider a Covid test that has 3% false positive and 0% false negative probability, where under H_0 'the person does not have Covid' and under H_1 'the person has Covid'. If someone tests positive, what is the probability that the person does have Covid?

- [x] $\frac{1}{3.97}$
- [ ] $\frac{0.99}{4.96}$
- [ ] $\frac{1}{4.96}$
- [ ] $\frac{0.97}{0.99}$

Question 6 How many different ways are there to split 8 students into 4 groups of size 2?

- [ ] 2520
- [x] 105
- [ ] 1680
- [ ] 840

Question 7 Let $X_1, X_3, X_5, X_7, \ldots$ be a sequence of i.i.d. Poisson(2) random variables and define $X_{2i} = X_{2i-1}$ for $i \in \mathbb{N}$. What does $\frac{1}{n}(X_1^2 + X_2^2 + \cdots + X_n^2)$ converge to (in probability)?

- [ ] 2
- [ ] 8
- [ ] 4
- [x] 6

Second part: true/false questions

For each question, mark the box TRUE if the statement is **always true** and the box FALSE if it is **not always true** (i.e., it is sometimes false).

Question 8 Consider random variables $X, Y \sim \mathcal{N}_2((0 \quad 0)^T, \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix})$ having a multinomial Gaussian distribution. For a matrix $A \in \mathbb{R}^{2\times 2}$ consider the linear transformation of $\begin{pmatrix} X' \\ Y' \end{pmatrix} = A \begin{pmatrix} X \\ Y \end{pmatrix}$.
Claim: There is no other matrix other than $A = I$ (identity matrix) such that X', Y' are independent and each of them are standard normal variables, i.e., $X', Y' \sim \mathcal{N}(0, 1)$.

☐ TRUE ■ FALSE

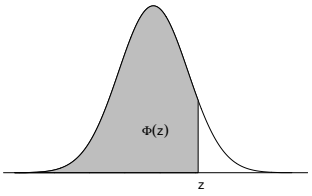
Question 9 A fair dice is thrown twice independently. Let D_1, D_2 be the values of the dice in the first and second throw.
Claim: the events $E_1 : D_1 = 2$ and $E_2 : D_1 + D_2 = 7$ are independent.

■ TRUE ☐ FALSE

Distributions

Distribution	PMF/PDF	Expected Value	Variance	MGF
Bernoulli Bern(p)	$P(X=1)=p$ $P(X=0)=1-p$	p	$p(1-p)$	$1-p+pe^t$
Binomial Bin(n,p)	$P(X=k)=\binom{n}{k}p^k(1-p)^{n-k}$ $k\in\{0,1,2,\dots n\}$	np	$np(1-p)$	$(1-p+pe^t)^n$
Geometric Geom(p)	$P(X=k)=(1-p)^{k-1}p$ $k\in\{1,2,\dots\}$	$\frac{1}{p}$	$\frac{1-p}{p^2}$	$\frac{pe^t}{1-(1-p)e^t}$ $(1-p)e^t<1$
Neg. Binom. NegBin(r,p)	$P(X=x)=\binom{x-1}{r-1}p^r(1-p)^{x-r}$ $x\in\{r,r+1,r+2,\dots\}$	$\frac{r}{p}$	$\frac{r(1-p)}{p^2}$	$(\frac{pe^t}{1-(1-p)e^t})^r$ $(1-p)e^t<1$
Hypergeom. HypG(w,b,n)	$P(X=k)=\frac{\binom{w}{k}\binom{b}{n-k}}{\binom{w+b}{n}}$ $k\in\{0,1,2,\dots,n\}$	$\mu=\frac{nw}{b+w}$	$\left(\frac{w+b-n}{w+b-1}\right)n\frac{\mu}{n}(1-\frac{\mu}{n})$	messy
Poisson Pois(λ)	$P(X=k)=\frac{e^{-\lambda}\lambda^k}{k!}$ $k\in\{0,1,2,\dots\}$	λ	λ	$e^{\lambda(e^t-1)}$
Uniform U(a,b)	$f(x)=\frac{1}{b-a}$ $x\in[a,b]$	$\frac{a+b}{2}$	$\frac{(b-a)^2}{12}$	$\frac{e^{tb}-e^{ta}}{t(b-a)}$
Exponential exp(λ)	$f(x)=\lambda e^{-\lambda x}$ $x\in(0,\infty)$	$\frac{1}{\lambda}$	$\frac{1}{\lambda^2}$	$\frac{\lambda}{\lambda-t}$, $t<\lambda$
Normal $\mathcal{N}(\mu,\sigma^2)$	$f(x)=\frac{1}{\sigma\sqrt{2\pi}}e^{-\frac{(x-\mu)^2}{2\sigma^2}}$ $x\in(-\infty,\infty)$	μ	σ^2	$e^{t\mu+\frac{\sigma^2t^2}{2}}$
Chi-Square χ^2_n	$\frac{1}{2^{n/2}\Gamma(n/2)}x^{n/2-1}e^{-x/2}$ $x\in(0,\infty)$	n	$2n$	$(1-2t)^{-n/2}$ $t<1/2$

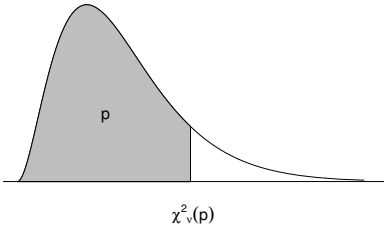
Standard normal distribution $\Phi(z)$



For $z < 0$ we use symmetry: $\mathbb{P}(Z \leq z) = \Phi(z) = 1 - \Phi(-z)$, $z \in \mathbb{R}$.

z	0	1	2	3	4	5	6	7	8	9
0.0	.50000	.50399	.50798	.51197	.51595	.51994	.52392	.52790	.53188	.53586
0.1	.53983	.54380	.54776	.55172	.55567	.55962	.56356	.56750	.57142	.57535
0.2	.57926	.58317	.58706	.59095	.59483	.59871	.60257	.60642	.61026	.61409
0.3	.61791	.62172	.62552	.62930	.63307	.63683	.64058	.64431	.64803	.65173
0.4	.65542	.65910	.66276	.66640	.67003	.67364	.67724	.68082	.68439	.68793
0.5	.69146	.69497	.69847	.70194	.70540	.70884	.71226	.71566	.71904	.72240
0.6	.72575	.72907	.73237	.73565	.73891	.74215	.74537	.74857	.75175	.75490
0.7	.75804	.76115	.76424	.76730	.77035	.77337	.77637	.77935	.78230	.78524
0.8	.78814	.79103	.79389	.79673	.79955	.80234	.80511	.80785	.81057	.81327
0.9	.81594	.81859	.82121	.82381	.82639	.82894	.83147	.83398	.83646	.83891
1.0	.84134	.84375	.84614	.84850	.85083	.85314	.85543	.85769	.85993	.86214
1.1	.86433	.86650	.86864	.87076	.87286	.87493	.87698	.87900	.88100	.88298
1.2	.88493	.88686	.88877	.89065	.89251	.89435	.89617	.89796	.89973	.90147
1.3	.90320	.90490	.90658	.90824	.90988	.91149	.91309	.91466	.91621	.91774
1.4	.91924	.92073	.92220	.92364	.92507	.92647	.92786	.92922	.93056	.93189
1.5	.93319	.93448	.93574	.93699	.93822	.93943	.94062	.94179	.94295	.94408
1.6	.94520	.94630	.94738	.94845	.94950	.95053	.95154	.95254	.95352	.95449
1.7	.95543	.95637	.95728	.95818	.95907	.95994	.96080	.96164	.96246	.96327
1.8	.96407	.96485	.96562	.96638	.96712	.96784	.96856	.96926	.96995	.97062
1.9	.97128	.97193	.97257	.97320	.97381	.97441	.97500	.97558	.97615	.97670
2.0	.97725	.97778	.97831	.97882	.97932	.97982	.98030	.98077	.98124	.98169
2.1	.98214	.98257	.98300	.98341	.98382	.98422	.98461	.98500	.98537	.98574
2.2	.98610	.98645	.98679	.98713	.98745	.98778	.98809	.98840	.98870	.98899
2.3	.98928	.98956	.98983	.99010	.99036	.99061	.99086	.99111	.99134	.99158
2.4	.99180	.99202	.99224	.99245	.99266	.99286	.99305	.99324	.99343	.99361
2.5	.99379	.99396	.99413	.99430	.99446	.99461	.99477	.99492	.99506	.99520
2.6	.99534	.99547	.99560	.99573	.99585	.99598	.99609	.99621	.99632	.99643
2.7	.99653	.99664	.99674	.99683	.99693	.99702	.99711	.99720	.99728	.99736
2.8	.99744	.99752	.99760	.99767	.99774	.99781	.99788	.99795	.99801	.99807
2.9	.99813	.99819	.99825	.99831	.99836	.99841	.99846	.99851	.99856	.99861
3.0	.99865	.99869	.99874	.99878	.99882	.99886	.99889	.99893	.99896	.99900
3.1	.99903	.99906	.99910	.99913	.99916	.99918	.99921	.99924	.99926	.99929
3.2	.99931	.99934	.99936	.99938	.99940	.99942	.99944	.99946	.99948	.99950
3.3	.99952	.99953	.99955	.99957	.99958	.99960	.99961	.99962	.99964	.99965
3.4	.99966	.99968	.99969	.99970	.99971	.99972	.99973	.99974	.99975	.99976
3.5	.99977	.99978	.99978	.99979	.99980	.99981	.99981	.99982	.99983	.99983
3.6	.99984	.99985	.99985	.99986	.99986	.99987	.99987	.99988	.99988	.99989
3.7	.99989	.99990	.99990	.99990	.99991	.99991	.99992	.99992	.99992	.99992
3.8	.99993	.99993	.99993	.99994	.99994	.99994	.99994	.99995	.99995	.99995
3.9	.99995	.99995	.99996	.99996	.99996	.99996	.99996	.99996	.99997	.99997

χ^2_ν distribution



$\chi^2_\nu(p)$: quantiles for the chi-square distribution with ν degrees of freedom.

ν	.005	.01	.025	.05	.10	.25	.50	.75	.90	.95	.975	.99	.995	.999
1	0	.0002	.010	.0039	.0158	.102	.455	1.32	2.71	3.84	5.02	6.63	7.88	10.8
2	.0100	.0201	.0506	.103	.211	.575	1.39	2.77	4.61	5.99	7.38	9.21	10.6	13.8
3	.0717	.115	.216	.352	.584	1.21	2.37	4.11	6.25	7.81	9.35	11.3	12.8	16.3
4	.207	.297	.484	.711	1.06	1.92	3.36	5.39	7.78	9.49	11.1	13.3	14.9	18.5
5	.412	.554	.831	1.15	1.61	2.67	4.35	6.63	9.24	11.1	12.8	15.1	16.7	20.5
6	.676	.872	1.24	1.64	2.20	3.45	5.35	7.84	10.6	12.6	14.4	16.8	18.5	22.5
7	.989	1.24	1.69	2.17	2.83	4.25	6.35	9.04	12.0	14.1	16.0	18.5	20.3	24.3
8	1.34	1.65	2.18	2.73	3.49	5.07	7.34	10.2	13.4	15.5	17.5	20.1	22.0	26.1
9	1.73	2.09	2.70	3.33	4.17	5.90	8.34	11.4	14.7	16.9	19.0	21.7	23.6	27.9
10	2.16	2.56	3.25	3.94	4.87	6.74	9.34	12.5	16.0	18.3	20.5	23.2	25.2	29.6
11	2.60	3.05	3.82	4.57	5.58	7.58	10.3	13.7	17.3	19.7	21.9	24.7	26.8	31.3
12	3.07	3.57	4.40	5.23	6.30	8.44	11.3	14.8	18.5	21.0	23.3	26.2	28.3	32.9
13	3.57	4.11	5.01	5.89	7.04	9.30	12.3	16.0	19.8	22.4	24.7	27.7	29.8	34.5
14	4.07	4.66	5.63	6.57	7.79	10.2	13.3	17.1	21.1	23.7	26.1	29.1	31.3	36.1
15	4.60	5.23	6.26	7.26	8.55	11.0	14.3	18.2	22.3	25.0	27.5	30.6	32.8	37.7
16	5.14	5.81	6.91	7.96	9.31	11.9	15.3	19.4	23.5	26.3	28.8	32.0	34.3	39.3
17	5.70	6.41	7.56	8.67	10.1	12.8	16.3	20.5	24.8	27.6	30.2	33.4	35.7	40.8
18	6.26	7.01	8.23	9.39	10.9	13.7	17.3	21.6	26.0	28.9	31.5	34.8	37.2	42.3
19	6.84	7.63	8.91	10.1	11.7	14.6	18.3	22.7	27.2	30.1	32.9	36.2	38.6	43.8
20	7.43	8.26	9.59	10.9	12.4	15.5	19.3	23.8	28.4	31.4	34.2	37.6	40.0	45.3
21	8.03	8.90	10.3	11.6	13.2	16.3	20.3	24.9	29.6	32.7	35.5	38.9	41.4	46.8
22	8.64	9.54	11.0	12.3	14.0	17.2	21.3	26.0	30.8	33.9	36.8	40.3	42.8	48.3
23	9.26	10.2	11.7	13.1	14.8	18.1	22.3	27.1	32.0	35.2	38.1	41.6	44.2	49.7
24	9.89	10.9	12.4	13.8	15.7	19.0	23.3	28.2	33.2	36.4	39.4	43.0	45.6	51.2
25	10.5	11.5	13.1	14.6	16.5	19.9	24.3	29.3	34.4	37.7	40.6	44.3	46.9	52.6
26	11.2	12.2	13.8	15.4	17.3	20.8	25.3	30.4	35.6	38.9	41.9	45.6	48.3	54.1
27	11.8	12.9	14.6	16.2	18.1	21.7	26.3	31.5	36.7	40.1	43.2	47.0	49.6	55.5
28	12.5	13.6	15.3	16.9	18.9	22.7	27.3	32.6	37.9	41.3	44.5	48.3	51.0	56.9
29	13.1	14.3	16.0	17.7	19.8	23.6	28.3	33.7	39.1	42.6	45.7	49.6	52.3	58.3
30	13.8	15.0	16.8	18.5	20.6	24.5	29.3	34.8	40.3	43.8	47.0	50.9	53.7	59.7
40	20.7	22.2	24.4	26.5	29.1	33.7	39.3	45.6	51.8	55.8	59.3	63.7	66.8	73.4
50	28.0	29.7	32.4	34.8	37.7	42.9	49.3	56.3	63.2	67.5	71.4	76.2	79.5	86.7
60	35.5	37.5	40.5	43.2	46.5	52.3	59.3	67.0	74.4	79.1	83.3	88.4	92.0	99.6
70	43.3	45.4	48.8	51.7	55.3	61.7	69.3	77.6	85.5	90.5	95.0	100.	104.	112.